

Gravitational Thermodynamics of Space-time Foam in One-loop Approximation

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Abstract

We show from one-loop quantum gravity and statistical thermodynamics that the thermodynamics of quantum foam in flat space-time and Schwarzschild space-time is exactly the same as that of Hawking-Unruh radiation in thermal equilibrium. This means we show unambiguously that Hawking-Unruh thermal radiation should contain thermal gravitons or the contribution of quantum space-time foam. As a by-product, we give also the quantum gravity correction in one-loop approximation to the classical black hole thermodynamics.

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The space-time foam-like structure (FLS) was firstly proposed by J. A. Wheeler about forty years ago[1]. He argued that the space-time may have a multiple connected non-trivial topological structure in Planck scale, though it seems smooth and simply connected in the large. The possible influence of FLS on field theory and the thermodynamical properties of FLS itself have been discussed by many authors[2-9]. In this paper, we would like to discuss the thermodynamical properties of FLS only from one-loop quantum gravity and statistical thermodynamics, so the result we get may be much more reliable.

If time in Euclidean quantum gravity has an imaginary period of $i\beta = i(1/T)$, (henceforth, we take $\hbar = c = G = k = 1$) then the partition function

$$Z = \sum_n \exp(-\beta E_n) \quad (1)$$

of canonical ensemble can be rewritten as an Euclidean path integral

$$Z = \int D(g, \phi) \exp(-\hat{I}(g, \phi)), \quad (2)$$

where $\hat{I}(g, \phi)$ is the Euclideanized action of gravity, g , and matter field, ϕ , and E_n is the n -th energy eigenvalue of certain differential field operator on its eigenstate vector $|g, \phi\rangle_n$. For the pure quantum gravity case, we put $\phi = 0$ and

$$g_{ab} = g_{ab}^{(0)} + \bar{g}_{ab}, \quad (3)$$

where \bar{g}_{ab} is the metric fluctuation of the background metric $g_{ab}^{(0)}$, then we can expand the action in Taylor series about the background field $g^{(0)}$ as

$$\hat{I}(g) = \hat{I}(g^{(0)}) + \hat{I}_2(\bar{g}) + [\text{higher-order terms}], \quad (4)$$

where $\hat{I}_2(\bar{g})$ is the well known one-loop term of the Euclidean gravitational action. In one-loop approximation, the logarithm of partition function, Z , reads

$$\ln Z = -\hat{I}(g^{(0)}) + \ln \int D(\bar{g}) \exp(-\hat{I}_2(\bar{g})). \quad (5)$$

As $\hat{I}(g^{(0)})$ is equal to the Gibbons-Hawking's surface term for vacuum Einstein gravity without cosmological term, the contributions to $\ln Z$ come from the surface term and $\hat{I}_2(\bar{g})$ in one-loop approximation.

Now from quantum gravity and statistical thermodynamics we try to study the gravitational thermodynamics of the one-loop quantum gravity for flat space-time and Schwarzschild space-time background. Let us consider the gravitational field inside volume V and with an imaginary time period $i\beta$. Hawking showed exactly that $\ln Z$ in one-loop approximation with flat space-time background reads[10]

$$\ln Z = \frac{4\pi^3 r_0^3 T^3}{135} = \frac{\pi^2}{45} \beta^{-3} V \quad (6)$$

for a system at temperature $T = \beta^{-1}$, contained in a spherical box of radius r_0 , where the Casimir effect of the finite size of volume V is neglected. (Note that the factor $4\pi^5$ in Eq.(15.99) of Hawking's original paper [10] should be corrected as $4\pi^3$ in Eq.(6).) Hawking argued that Eq.(6) is just the contribution of the thermal gravitons to the partition function. However, in our opinion, if FLS is created from metric fluctuation, an equivalent interpretation of Eq.(6) as the contribution of FLS can also be given. Let P_n be the probability of FLS in volume V in the n -th energy eigenstate, then from

$$S = - \sum_n P_n \ln P_n \quad (7)$$

and

$$P_n = Z^{-1} \exp(-\beta E_n), \quad (8)$$

the entropy of FLS in V is given by

$$S = \beta \langle E \rangle + \ln Z, \quad (9)$$

where the expected value of energy is given by

$$\langle E \rangle = - \frac{\partial}{\partial \beta} \ln Z. \quad (10)$$

From Eqs. (6), (9) and (10) it is easy to show that the entropy, S , and energy, U , of FLS inside volume V are respectively

$$S = \frac{4\pi^2}{45} \beta^{-3} V \quad (11)$$

and

$$U = \langle E \rangle = \frac{\pi^2}{15} \beta^{-4} V. \quad (12)$$

So, the entropy density, ρ_S , and energy density, ρ_U , of FLS are respectively

$$\rho_S = \frac{4\pi^2}{45} T^3 \quad (13)$$

and

$$\rho_U = \frac{\pi^2}{15} T^4. \quad (14)$$

They are exactly the same as that of the black body radiation. As is known, the time coordinate of inertial system in flat space-time has no imaginary period, or the same, the period β in inertial system is infinite. Hence the temperature of FLS is always zero for a inertial observer. However, the only case that the time coordinate in flat space-time has a finite imaginary period is that of Rindler system. This is clear from the coordinate transformation from inertial coordinates (t, x, θ, φ) to Rindler coordinates $(\eta, \xi, \theta, \varphi)$,

$$\begin{aligned} t &= a^{-1} e^{a\xi} sh(a\eta) \\ x &= a^{-1} e^{a\xi} ch(a\eta) \end{aligned} \quad (15)$$

Evidently, η has an imaginary period of $2\pi/a$ after it is Euclideanized, i.e., $\eta \rightarrow -i\eta$. So, $\beta = 2\pi/a$ and the famous Unruh temperature[11,12]

$$T_U = \beta^{-1} = \frac{a}{2\pi} \quad (16)$$

results.

The above discussion suggests that, though the temperature of FLS for inertial observer is zero, but the Rindler observer will find himself immersed in a heat bath of temperature T_U of FLS. Now a problem confront us is whether we are really in an inertial system or in a Rindler system with heat bath? It seems no choice can be given a priori. The only reasonable choice depends on whether we can measure out the temperature of FLS or not. As is known, there is an universal background black body radiation of $\sim 3^0k$ everywhere in

the universe. If the idea of thermal gravitons or thermal FLS is not wrong, the densities ρ_S and ρ_U may be too small compared with that of the $\sim 3^0 k$ background radiation, so that a measurement of them can hardly be given, especially, if we remember that in order to get an Unruh temperature of $1^0 k$ the proper acceleration of the Rindler observer should approximately be [13]

$$\alpha^{-1} = ae^{-a\xi} \cong 2.4 \times 10^{20} m/s^2 \cong 10^{19} g_E, \quad (17)$$

where g_E is the proper acceleration on the surface of the earth. Hence, it seems highly impossible that FLS can have any measurable thermal properties in practice.

Hawking also showed that the one-loop approximation of the logarithm of the partition function, $\ln Z$, for pure gravity in Schwarzschild black hole background metric is given approximately by [10]

$$\ln Z = -\hat{I}(g^{(0)}) + \ln \int D(\bar{g}) \exp(-\hat{I}_2(\bar{g})) = -\frac{\beta^2}{16\pi} + \frac{106}{45} \ln\left(\frac{\beta}{\beta_0}\right) + \frac{4\pi^3 r_0^3}{135\beta^3} + O(r_0^2 \beta^{-2}), \quad (18)$$

where $\hat{I}(g^{(0)}) = \beta^2/16\pi$ is the Gibbons-Hawking's surface term of Schwarzschild space-time, $\beta = T^{-1} = 8\pi M$ is just the imaginary time period of Schwarzschild black hole, β_0 is an arbitrary constant of energy dimensionality, and r_0 is the proper radius of a spherical box enclosing the Euclidean section of a Schwarzschild black hole in its center. From Eqs. (9) (10) and (18), we can get the gravitational entropy, S , and energy, U , of the whole system of proper volume V as

$$U = -\frac{\partial}{\partial \beta} \ln Z = M - \frac{53}{180\pi M} + \frac{\pi^2}{15} T^4 V, \quad (19)$$

$$S = \beta < E > + \ln Z = 4\pi M^2 + \frac{106}{45} \left[\ln\left(\frac{M}{M_0}\right) - 1 \right] + \frac{4\pi^2}{45} T^3 V, \quad (20)$$

where $M_0 \equiv \beta_0/8\pi$.

The last terms in the right-hand sides of Eqs. (19) and (20) are exactly the same as that of the Hawking radiation in equilibrium, that can only be ascribed to the contribution of thermal gravitons or of quantum foam, while the first terms are the familiar contribution

of classical Schwarzschild black hole, and the second terms which originate from one-loop quantum gravity are no doubt the quantum gravity correction to the classical Schwarzschild black hole. It is interesting to note that, the quantum gravity correction of black hole entropy is always negative and inverse proportional to the black hole mass, while the quantum gravity correction of black hole entropy is negative, positive or zero in the case $\frac{M}{M_0} < e$, $> e$, or $= e$.

In summary, the gravitational thermodynamics of FLS in one-loop approximation is exactly the same as that of Hawking-Unruh radiation in thermal equilibrium. In other words, we show unambiguously that the Hawking-Unruh thermal radiation not only contain the matter particles but also contain the thermal gravitons or the contribution from the quantum FLS of space-time. We show also that the one-loop quantum gravity gives quantum correction to the classical thermodynamics of Schwarzschild black hole.

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